Handout 2 (Spring 2020 term)

1. Within the theory of a free massless real scalar field $\varphi$ in $D = 1 + 1$, consider the operator-valued distribution describing the normal-ordered energy density
   
   $:$ $\hat{\rho}[f] : = \frac{1}{2} \int_{\mathbb{R}} : (\partial_t \varphi(x,0))^2 + (\partial_x \varphi(x,0))^2 : f(x) dx.$
   
   This distribution takes a non-negative $C^\infty$ test function $f : \mathbb{R} \to \mathbb{R} + \{0\}$ with compact support and returns a hermitian operator. Find the norm of the state $|\Phi_\alpha\rangle$ in terms of the Fourier image of $f$. Show that there exists an $\alpha \in \mathbb{C}$ and a test function $f$ such that the normal-ordered energy density operator is not positive-definite, which is the price we paid for a zero expectation over the vacuum state.
   
   $N.B.\: This\: means\: that\: the\: normal-ordered\: energy\: density\: operator$\: is\: not\: positive-definite,\: which\: is\: the\: price\: we\: paid\: for\: a\: zero\: expectation\: over\: the\: vacuum\: state.

2. Starting from a regularized expression for the stress tensor $\langle T_{\mu\nu}(x) \rangle$ of a real massless scalar field in the presence of a moving mirror in 1D (see slide 8/13 in lecture 7), check the expression for the renormalized stress tensor
   
   $\langle T_{\text{ren}}^{00} \rangle = -\langle T_{\text{ren}}^{01} \rangle = \frac{1}{12\pi} \sqrt{\rho'(x_-)} \left[ \frac{1}{\sqrt{\rho'(x_-)}} \right].$

3. Representing the effect of the external magnetic field on the photon dispersion by a regularization, find the 'phase volume' of a photon splitting process into two final photons
   
   $V_{\text{ph}}^{(2)}(k) = \lim_{\epsilon \to +0} \int d^3k_1 d^3k_2 \delta^3(k - k_1 - k_2) \delta(\|k\| - |k_1| - |k_2| + \epsilon).$

4. Using the well-known expression for the Heisenberg–Euler effective action (see, e.g., lecture 9), find the term in this Lagrangian that is responsible for the photon splitting in a strong external magnetic field $B$ (specifically, for the contribution of the hexagon diagram to the amplitude, slide 10/13).

5. Within a scalar $D = 1 + 1$ theory with the Lagrangian
   
   $L = \frac{1}{2} (\partial_{\mu}\varphi)^2 - \frac{\lambda}{4} (\varphi^2 - v^2)^2, \quad m \equiv v \sqrt{\lambda},$
   
   find the mass counterterm $\delta L = -\frac{1}{2} \delta m^2 \varphi^2$ which renormalizes the self-energy in the leading order in $\lambda$ (see slide 17/21 in lecture 8). In the loop integrals, use the momentum cutoff $|k| \leq \Lambda m \to \infty$, where $k \equiv km$ is the spatial part of the loop momentum.

6. Prove the Fierz identities ($\varphi, \chi, \psi, \omega$ are classical, anticommuting spinor fields)
   
   $\bar{\varphi} \gamma_{\mu}^\nu \bar{\psi} \gamma_{\mu} \lambda \omega = \bar{\varphi} \gamma_{\mu}^\nu \bar{\psi} \gamma_{\mu} \lambda \chi, \quad \bar{\varphi} \gamma_{\mu}^\nu \bar{\psi} \gamma_{\mu} \lambda \omega = -2 \bar{\varphi} P_{L,R} \bar{\psi} P_{R,L} \chi,$
   
   where $P_{L,R} \equiv (1 \mp \gamma_5)/2, \gamma_{L,R}^\mu \equiv \gamma^\mu P_{L,R}$.

7. Starting with a general expression for the 1-loop effective potential $V^{(1)}(\eta_E^2)$ for axions in the $D = 3 + 1$ Axion–Wess–Zumino model (see slide 8/14 in lecture 11), use the momentum cutoff regularization $|p_E| \leq \Lambda \to \infty$ and find the divergent part of the potential.