Faculty of Physics, Lomonosov Moscow State University

Advanced Quantum Field Theory: Modern Applications in HEP, Astro & Cond-Mat
Instructor: O. Kharlanov

Handout 2 (Spring 2020 term)

1. Within the theory of a free massless real scalar field φ in $D = 1 + 1$, consider the operator-valued distribution describing the normal-ordered energy density

$$:\hat{\rho}[f]: = \frac{1}{2}\int_{\mathbb{R}} :(\partial_t\hat{\varphi}(x,0))^2 + (\partial_x\hat{\varphi}(x,0))^2: \, f(x)\mathrm{d}x.$$

This distribution takes a non-negative C^∞ test function $f : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ with compact support and returns a hermitian operator. Find the norm of the state $|\Xi\rangle = :\hat{\rho}[f]: \, |0\rangle$ in terms of the Fourier image of f. Show that there exists an $\alpha \in \mathbb{C}$ and a test function f such that the normal-ordered energy density expectation in a superposition state $|\Phi_\alpha\rangle$ is negative

$$\frac{\langle\Phi_\alpha| \, :\hat{\rho}[f]: \, |\Phi_\alpha\rangle}{\int_{\mathbb{R}} f(x)\mathrm{d}x} < 0, \qquad |\Phi_\alpha\rangle \equiv \frac{|0\rangle + \alpha\,|\Xi\rangle}{\sqrt{1 + |\alpha|^2\,\langle\Xi|\Xi\rangle}}.$$

N.B. This means that the normal-ordered energy density operator is not positive-definite, which is the price we paid for a zero expectation over the vacuum state.

2. Starting from a regularized expression for the stress tensor $\langle T_{\mu\nu}(x_-)\rangle$ of a real massless scalar field in the presence of a moving mirror in 1D (see slide 8/13 in lecture 7), check the expression for the renormalized stress tensor

$$\langle T^{\mathrm{ren}}_{00}\rangle = -\langle T^{\mathrm{ren}}_{01}\rangle = \frac{1}{12\pi}\sqrt{p'(x_-)}\left[\frac{1}{\sqrt{p'(x_-)}}\right]''.$$

3. Representing the effect of the external magnetic field on the photon dispersion by a regularization, find the 'phase volume' of a photon splitting process into two final photons

$$V^{(2)}_{\mathrm{ph}}(\boldsymbol{k}) = \lim_{\epsilon\to+0}\int \mathrm{d}^3k_1\mathrm{d}^3k_2\;\delta^3(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\,\delta(|\boldsymbol{k}| - |\boldsymbol{k}_1| - |\boldsymbol{k}_2| + \epsilon).$$

4. Using the well-known expression for the Heisenberg–Euler effective action (see, e.g., lecture 9), find the term in this Lagrangian that is responsible for the photon splitting in a strong external magnetic field $\bar{\boldsymbol{B}}$ (specifically, for the contribution of the hexagon diagram to the amplitude, slide 10/13).

5. Within a scalar $D = 1 + 1$ theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\lambda}{4}(\varphi^2 - v^2)^2, \qquad m \equiv v\sqrt{\lambda},$$

find the mass counterterm $\delta\mathcal{L} = -\frac{1}{2}\delta m^2\varphi^2$ which renormalizes the self-energy in the leading order in λ (see slide 17/21 in lecture 8). In the loop integrals, use the momentum cutoff $|k| \leq \Lambda m \to \infty$, where $k \equiv \kappa m$ is the spatial part of the loop momentum.

6. Prove the Fierz identities ($\varphi, \chi, \psi, \omega$ are classical, anticommuting spinor fields)

$$\bar{\varphi}\gamma^\mu_{\mathrm{L}}\chi\;\bar{\psi}\gamma_{\mu\mathrm{L}}\omega = \bar{\varphi}\gamma^\mu_{\mathrm{L}}\omega\;\bar{\psi}\gamma_{\mu\mathrm{L}}\chi, \qquad \bar{\varphi}\gamma^\mu_{\mathrm{R}}\chi\;\bar{\psi}\gamma_{\mu\mathrm{L}}\omega = -2\bar{\varphi}P_{\mathrm{L}}\omega\;\bar{\psi}P_{\mathrm{R}}\chi,$$

where $P_{\mathrm{L,R}} \equiv (1 \mp \gamma_5)/2$, $\gamma^\mu_{\mathrm{L,R}} \equiv \gamma^\mu P_{\mathrm{L,R}}$.

7. Starting with a general expression for the 1-loop effective potential $V^{(1)}(\eta_E^2)$ for axions in the $D = 3+1$ Axion–Wess–Zumino model (see slide 8/14 in lecture 11), use the momentum cutoff regularization $|p_E| \leq \Lambda \to \infty$ and find the divergent part of the potential.