Faculty of Physics, Lomonosov Moscow State University

Advanced Quantum Field Theory: Modern Applications in HEP, Astro & Cond-Mat
Instructor: O. Kharlanov

Handout 1 (Spring 2020 term)

1. Find the electric field of an isolated hydrogen atom in the ground state $1s$.

2. Starting from a Hamiltonian $\hat{H}(\boldsymbol{X}_1, \boldsymbol{X}_2)$ of two hydrogen atoms centered at points $\boldsymbol{X}_{1,2}$ and separated by distance $R = |\boldsymbol{X}_2 - \boldsymbol{X}_1| \gg r_{\text{Bohr}}$, show that the interaction of the two atoms reads
$$\hat{H}_{\text{int}} = \frac{(\hat{\boldsymbol{d}}_1 \hat{\boldsymbol{d}}_2) - 3(\hat{\boldsymbol{d}}_1 \boldsymbol{\rho})(\hat{\boldsymbol{d}}_2 \boldsymbol{\rho})}{R^3} + o(1/R^3),$$
where the dipole moment operators $\hat{\boldsymbol{d}}_a = e(\hat{\boldsymbol{x}}_a - \boldsymbol{X}_a)$ and $\boldsymbol{\rho} = \frac{\boldsymbol{X}_2 - \boldsymbol{X}_1}{|\boldsymbol{X}_2 - \boldsymbol{X}_1|} \equiv \frac{\boldsymbol{R}}{R}$.

3. The three states $2p_{x,y,z}$ are defined as the eigenstates of $\hat{l}_{x,y,z}$, respectively, with the zero eigenvalue. Show that for a hydrogen atom,
$$\langle 2s | \, x_b \, | 1s \rangle = 0, \quad \langle 2p_a | \, x_b \, | 1s \rangle = \Delta \delta_{ab}, \qquad a, b = x, y, z,$$
and find the complex coefficient Δ.

4. Find the second-quantized expression for the field operator $\hat{\phi}(\boldsymbol{x}, t)$ of a massless scalar field in a compact spatial domain $\boldsymbol{x} \in D$, in terms of the eigenfunctions of $(-\nabla^2)$ in D. Find the second-quantized expression for the Hamiltonian of ϕ in this domain. Assume that the field satisfies Dirichlet boundary conditions on ∂D.

5. Find the Casimir energy using the exponential smooth cutoff + explicit renormalization for a massless scalar field in $D = 1 + 1$ (analogously to the $D = 3 + 1$ case studied in lecture 3). Compare your result with the one provided by the zeta function.

6. Find the Fourier image $g(z, z'; \omega, \boldsymbol{k}_\parallel)$ of the Green's function for a massless scalar field for $z, z' < 0$ (see lecture 4). The plate with Dirichlet boundary conditions lies in the $z = 0$ plane.

7. Find the system of PDEs on the dyadic Green's function $\Gamma'_{ia}(\boldsymbol{x}, \boldsymbol{x}'; \omega)$ (see slide 5/29 in lecture 5).

8. Show that any (smooth enough) transversal vector function $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{C}^3$, $\operatorname{div} \boldsymbol{f}(\boldsymbol{x}) = 0$, can be expressed in terms of a series over vector spherical harmonics
$$\boldsymbol{f}(\boldsymbol{x}) = \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \left\{ f_{\text{E}}^{jm}(r) \boldsymbol{X}_{jm}(\boldsymbol{n}) + \operatorname{rot}(f_{\text{M}}^{jm}(r) \boldsymbol{X}_{jm}(\boldsymbol{n})) \right\}, \quad \boldsymbol{X}_{jm}(\boldsymbol{n}) \equiv \frac{[\boldsymbol{x} \times \boldsymbol{\nabla}] Y_{jm}(\boldsymbol{n})}{\sqrt{j(j+1)}},$$
where the 'electric' and 'magnetic' coefficients of $\boldsymbol{f}$ are given by
$$f_{\text{E}}^{jm}(r) = \oint \boldsymbol{X}^*_{jm}(\boldsymbol{n}) \cdot \boldsymbol{f}(r\boldsymbol{n}) \, \mathrm{d}\Omega_{\boldsymbol{n}}, \quad f_{\text{M}}^{jm}(r) = -\frac{r}{\sqrt{j(j+1)}} \oint \boldsymbol{n} Y^*_{jm}(\boldsymbol{n}) \cdot \boldsymbol{f}(r\boldsymbol{n}) \, \mathrm{d}\Omega_{\boldsymbol{n}}.$$
Here, $r \equiv |\boldsymbol{x}|$, $r' \equiv |\boldsymbol{x}'|$, $\boldsymbol{n} \equiv \boldsymbol{x}/r$, $\boldsymbol{n}' \equiv \boldsymbol{x}'/r'$. *Hint: You can use Lemma 1 on slide 7/29 of lecture 5 as a starting point, without proof.*

9. Show that function $\Delta_{ia}(\boldsymbol{x}, \boldsymbol{x}') \equiv (\delta_{ia} \nabla^2 - \partial_i \partial_a) \delta^3(\boldsymbol{x} - \boldsymbol{x}')$ can be expressed in terms of a series over 'bivector' spherical harmonics
$$\Delta_{ia}(\boldsymbol{x}, \boldsymbol{x}') = \sum_{j,m} \left\{ \Delta_{\text{E}}^j(r, r') X^i_{jm}(\boldsymbol{n}) X^{a*}_{jm}(\boldsymbol{n}') + \epsilon_{ikl} \partial_k \epsilon_{abc} \partial'_b (\Delta_{\text{M}}^j(r, r') X^l_{jm}(\boldsymbol{n}) X^{c*}_{jm}(\boldsymbol{n}')) \right\},$$
$$\Delta_{\text{E}}^j(r, r') = -\left[\nabla_r^2 - \frac{j(j+1)}{r^2} \right] \frac{\delta(r - r')}{r^2}, \quad \Delta_{\text{M}}^j(r, r') = -\frac{\delta(r - r')}{r^3}.$$

Hint: check that Δ_{ia} possesses the necessary properties of function $\boldsymbol{f}$ in problem 8 in both indices/arguments, and then apply the expansion twice. Also, make sure Δ is a rotationally-invariant bivector field and use Schur's lemma.

10. Find the expression for the electromagnetic Casimir tension on a conducting sphere $r = R$, using the expression for the dyadic Green's function in the spherical basis,

$$f = -\frac{\mathrm{i}}{16\pi^2 R^2} \int\limits_{-\infty}^{+\infty} e^{-\mathrm{i}\omega\tau} \mathrm{d}\omega \sum_{j=1}^{\infty} (2j+1) \Big\{ \big(\omega^2 R^2 - j(j+1)\big) \Gamma'^{j}_{\mathrm{M}}(r, r'; \omega) + \frac{1}{\omega^2} \xi_r \xi_{r'} \Gamma'^{j}_{\mathrm{E}}(r, r'; \omega) \Big\}^{r,r' \to R-0}_{r,r' \to R+0},$$

where $\xi_r \equiv \partial_r (r\,\cdot\,)$ and $\tau \to +0$ is a point-splitting regulator.

11. For a massless real scalar field between two Dirichlet plates $z = 0$ and $z = vt$, $v \equiv \tanh\psi =$ const, renormalize the series expression for the field energy (see slide 6/9 in lecture 6). Compare the result for the renormalized force with the one shown in the lecture presentation and derive its quasi-static asymptotic

$$f_{\mathrm{Casimir}}(t) = -\frac{\pi^2}{480(vt)^4} \left\{ 1 + \frac{8}{3} v^2 + \dots \right\}, \qquad |v| \ll 1.$$