Faculty of Physics, Lomonosov Moscow State University

Advanced Quantum Field Theory: Modern Applications in HEP, Astro & Cond-Mat
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Handout 1 (Spring 2016 term)

1. Find the electric field of an isolated hydrogen atom in the ground state $1s$.

2. Show that the interaction of the two hydrogen atoms separated from each other by the distance $R \gg r_{\rm atomic}$, within the leading order in $r_{\rm atomic}/R$, is

$$H_{\rm int} \approx \frac{(\boldsymbol{d}_1\boldsymbol{d}_2) - 3(\boldsymbol{d}_1\boldsymbol{\rho})(\boldsymbol{d}_2\boldsymbol{\rho})}{R^3},$$

where the dipole moment operators $\boldsymbol{d}_a = e(\boldsymbol{x}_a - \boldsymbol{x}_a^{\rm nucleus})$ and $\boldsymbol{\rho} = \frac{\boldsymbol{x}_2^{\rm nucleus} - \boldsymbol{x}_1^{\rm nucleus}}{|\boldsymbol{x}_2^{\rm nucleus} - \boldsymbol{x}_1^{\rm nucleus}|} \equiv \frac{\boldsymbol{R}}{R}$.

3. The three states $2p_{x,y,z}$ are defined as the eigenstates of $l_{x,y,z}$, respectively, with the zero eigenvalue. Show that for a hydrogen atom,

$$\langle 2s|\, x_b\, |1s\rangle = 0, \quad \langle 2p_a|\, x_b\, |1s\rangle = \Delta\delta_{ab}, \qquad a, b = x, y, z,$$

and find the complex coefficient Δ.

4. Find the second-quantized expression for the field operator $\phi(x)$ of a massless scalar field in a compact spatial domain $\boldsymbol{x} \in D$, in terms of the eigenfunctions of $(-\nabla^2)$ in D. Find the second-quantized expression for the Hamiltonian of ϕ in this domain.

5. Find the Casimir energy using the exponential smooth cutoff + explicit renormalization for a massless scalar field in $D = 1 + 1$. Compare this result with the one given by the zeta function.

6. Find the Fourier image $g(z, z'; \omega, \boldsymbol{k}_\parallel)$ of the Green's function for a massless scalar field for $z, z' < 0$. The plate with Dirichlet boundary conditions lies in the $z = 0$ plane.

7. Find the system of PDEs for the dyadic Green's function $\Gamma'_{ia}(\boldsymbol{x}, \boldsymbol{x}'; \omega)$ (see Lect. 6).

8. Show that any (smooth enough) transversal vector function $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{C}^3$, $\operatorname{div} \boldsymbol{f}(\boldsymbol{x}) = 0$, can be expressed in terms of a series over vector spherical harmonics

$$\boldsymbol{f}(\boldsymbol{x}) = \sum_{j=1}^{\infty}\sum_{m=-j}^{j} \left\{ f_E^{jm}(r)\boldsymbol{X}_{jm}(\boldsymbol{n}) + \operatorname{rot}(f_M^{jm}(r)\boldsymbol{X}_{jm}(\boldsymbol{n}))\right\},$$

where the 'electric' and 'magnetic' parts of $\boldsymbol{f}$ are given by

$$f_E^{jm}(r) = \oint \boldsymbol{X}^*_{jm}(\boldsymbol{n}) \cdot \boldsymbol{f}(r\boldsymbol{n})\, d\Omega_{\boldsymbol{n}}, \quad f_M^{jm}(r) = -\frac{r}{\sqrt{j(j+1)}} \oint \boldsymbol{n} Y^*_{jm}(\boldsymbol{n}) \cdot \boldsymbol{f}(r\boldsymbol{n})\, d\Omega_{\boldsymbol{n}}.$$

Here, $\boldsymbol{n} \equiv \boldsymbol{x}/|\boldsymbol{x}|$, $\boldsymbol{n}' \equiv \boldsymbol{x}'/|\boldsymbol{x}'|$ are unit vectors and $\boldsymbol{X}_{jm}(\boldsymbol{n}) \equiv [\boldsymbol{x} \times \boldsymbol{\nabla}]Y_{jm}(\boldsymbol{n})/\sqrt{j(j+1)}$.

9. Show that function $\Delta_{ia}(\boldsymbol{x}, \boldsymbol{x}') \equiv (\delta_{ia}\nabla^2 - \partial_i\partial_a)\delta^3(\boldsymbol{x} - \boldsymbol{x}')$ can be expressed in terms of a series over 'bivector' spherical harmonics

$$\Delta_{ia}(\boldsymbol{x}, \boldsymbol{x}') = \sum_{j,m} \left\{ \Delta_E^j(r, r') X^i_{jm}(\boldsymbol{n}) X^{a*}_{jm}(\boldsymbol{n}') + \epsilon_{ikl}\partial_k\epsilon_{abc}\partial'_b(\Delta_M^j(r, r') X^l_{jm}(\boldsymbol{n}) X^{c*}_{jm}(\boldsymbol{n}'))\right\},$$

$$\Delta_E^j(r, r') = -\left[\nabla_r^2 - \frac{j(j+1)}{r^2}\right]\frac{\delta(r - r')}{r^2}, \quad \Delta_M^j(r, r') = -\frac{\delta(r - r')}{r^3}.$$

10. Find the expression for the electromagnetic Casimir tension on the sphere $r = R$, using the expression for the dyadic Green's function in the spherical basis,

$$f = -\frac{\mathrm{i}}{16\pi^2 R^2} \int\limits_{-\infty}^{+\infty} e^{-\mathrm{i}\omega\tau} \mathrm{d}\omega \sum_{j=1}^{\infty} (2j+1) \Big\{ \big(\omega^2 R^2 - j(j+1)\big) \Gamma'^j_M(r, r'; \omega) + \frac{1}{\omega^2} \xi_r \xi_{r'} \Gamma'^j_E(r, r'; \omega) \Big\}_{r,r' \to R+0}^{r,r' \to R-0},$$

where $\xi_r \equiv \partial_r (r\,\cdot\,)$ and $\tau \to +0$ is a point-splitting regulator.

11. For a massless scalar field between two Dirichlet plates $z = 0$ and $z = vt$, $v = \tanh\psi = \text{const}$, renormalize the series expression for the field energy (see Lect. 7). Compare the result with the one shown in the lecture presentation and check the asymptotic expression for the Casimir tension

$$f_{\text{Casimir}}(t) = -\frac{\pi^2}{480 (vt)^4} \left\{ 1 + \frac{8}{3} v^2 + \ldots \right\}, \qquad |v| \ll 1.$$